A Locus Example

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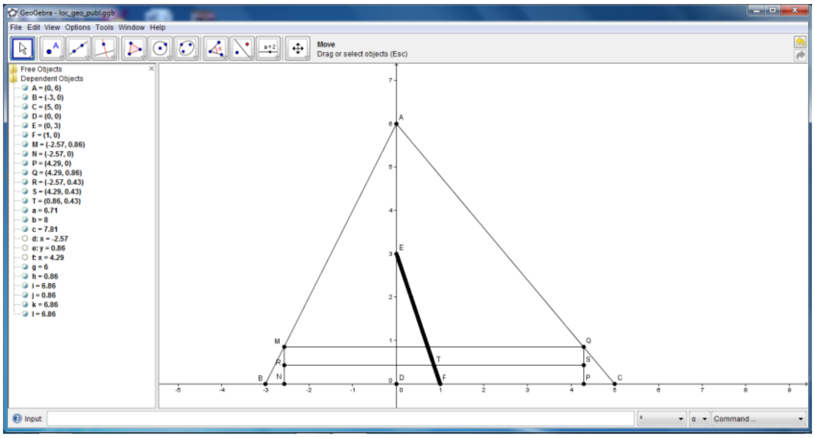
ABSTRACT *In the present paper I set out to exemplify the way in which GeoGebra can be used to solve a classic locus problem, namely the loci of the centers of symmetry of a rectangle with two vertices on one of the lines of a given triangle and the other two vertices on the other two lines. I will describe the way in which I created the GeoGebra file and then present two mathematical solutions: a synthetic (elementary) one and one using Cartesian coordinates.*

1. The GeoGebra task

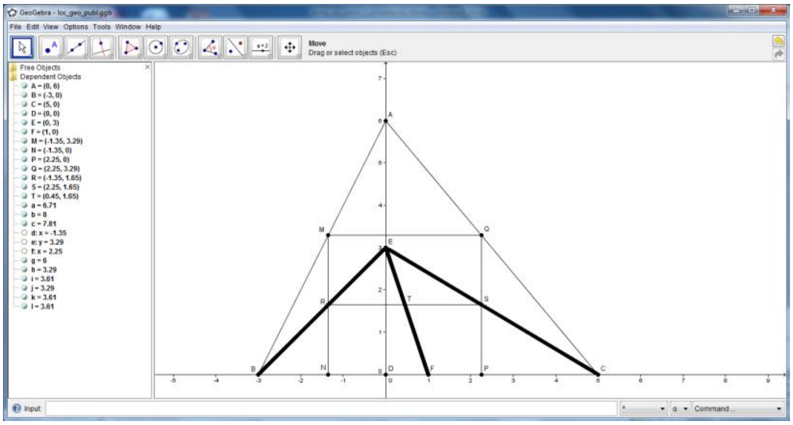
Create a GeoGebra file with which to visualize the loci of the centers of symmetry of the rectangles with two vertices on one of the lines of a given triangle and the other two vertices on the other two lines of the triangle.

2. The solution to the GeoGebra task

We build an *ABC* triangle. Preferably, although not very important, choose the *A* point on the *Oy* axis, the *B* and *C* points on the *Ox* axis with O situated between *B* and *C*. Build the *BC, AC* and *AB* line segments. Now choose a point on the *AB* line segment. This will be automatically named *D*. Rename it to *M*. Draw a parallel to the *Oy* axis from *M* building the point in which it intersects the *BC* line segment. Rename this point *N*. Draw the parallel to the *Ox* axis from *M* and determine the point in which it intersects the *AC* line segment. Rename this point *Q.* Draw a parallel to the *Oy* axis from *Q* and determine the point in which it intersects the *BC* line segment. Rename this point *P*. Now build the centers of the *MN* and *PQ* line segments and rename these points to *R* and *S* respectively. Build the centre of the *RS* line segment and rename this point *T*. choose the point in which the coordinates axes intersect. This point will be renamed D, build the AD line segment and its centre, named *E.* Finally, build the centre of the BC line segment. This will be named *F*. Hide the *MN, MQ, PQ* line segments*.* Activate the trace of the *T* point (right click on the point and left click on Trace On in the menu that appears). Move the *M* point to the *AB* line segment. You get the following drawing:



You would get an even more spectacular drawing by activating the traces of the *R* and *S* points.



3. The mathematical solution (elementary level)

When *M* describes the *AB* line segment, the *R* point (the centre of the *MN* line segment) describes the *BE* mean line of the *ABD* triangle. At the same time, *S* the centre of the *PQ* line segment describes the *CE* mean line of the *ACD* triangle. Therefore, the centre of symmetry *R* of the *MNPQ* rectangle describes the *EF* mean line of the *BEC* triangle. Thus, the points of the geometric locus are situated on the *EF* open line segment that unites the *E* centre of the height *AD* of the *ABC* triangle with the centre *F* of the *BC* line segment. Likewise: let there be *T* a random point between *E* and *F* and let there be *R, S* the points in which the parallel to *BC* through point *T* intersect the *BE* and *CE* line segments. Then *T* is the centre of the *RS* line segment. Let there be *M* and *N* the points in which the parallel drawn though *R* to *AD* intersect *AB* and *BC* respectively. The *R* is the centre of the *MN* line segment. Likewise, we draw through *S* the parallel to *AD* and name *P* and *Q* respectively the points in which this parallel intersects *BC* and *AC* respectively. Then *S* is the centre of the *PQ* line segment. From the above we deduce that *MNPQ* is a rectangle and *T* is its centre of symmetry. Therefore, the locus we are looking for is the *EF* open line segment.

4. The mathematical solution (using the analytic method)

In the following we will name x_U and y_U the coordinates to the arbitrary point U reported to a Cartesian system of coordinate axes. We make our calculations relatively to the above drawing in which we have $D(0{,}0), A(0{,}6), B(-3{,}0), C(5{,}0)$ and let there be $M(t,u)$ with $-3 < t < 0, 0 < u < 6.$ The equation of the AB line segment is

$$AB: y = 2x + 6$$

(You can verify this by building the AB line segment; its equation will be shown in the algebra window). Because M is on this line segment we deduce that $u = 2t + 6$. Then $y_Q = y_M = 2t + 6.$ because the equation of the AC line segment is

$$AC: y = -\frac{6}{5}x + 6$$

and Q is on this line segment, we deduce that $x_Q = -\frac{5t}{3}$. Considering that R, S are the centers of the MN and PQ line segments, we deduce $x_R = t,\ y_R = t + 3, x_S = -\frac{5t}{3}, y_S = t + 3.$ Therefore we have:

$$\begin{cases} x_T = -\dfrac{t}{3} \\ y_T = t + 3 \\ -3 < t < 0 \end{cases} \qquad (1)$$

From the condition $-3 < t < 0$ we deduce $0 < x_T < 1, 0 < y_T < 3.$ On the other hand we have $E(0{,}3)$ and $F(1{,}0)$. We can see from (1) that the locus for point T is the EF line segment without its extremities. You can verify the above results in the algebra window. This way you can emphasize the connection GeoGebra makes between geometric representations and algebraic calculations.

Now let us analyze the general case. We consider the Cartesian system of coordinate axes in which we have $D(0{,}0), A(0,h), B(b,0), C(c,0)$ and $M(t,u)$ cu $b < t < 0, 0 < u < h.$ The equation of the AB line segment is

$$AB: y = -\frac{h}{b}x + h.$$

Because M is on this line segment we deduce that $u = -\frac{h}{b}t + h$ and then:

$$y_Q = y_M = -\frac{h}{b}t + h.$$

The equation for the AC line segment is

$$AC: y = -\frac{h}{c}x + h.$$

Q is on this line segment and we deduce that $x_Q = -\frac{c}{b}t$. Therefore, we have:

$$x_R = t, y_R = -\frac{h}{2b}t + \frac{h}{2}, x_S = \frac{c}{b}t, y_S = -\frac{h}{2b}t + \frac{h}{2}.$$

As above, we get:

$$\begin{cases} x_T = \dfrac{b+c}{2b}t \\ y_T = -\dfrac{h}{2b}t + \dfrac{h}{2} \\ b < t < 0 \end{cases} \qquad (2)$$

From (2) we deduce that if $t \to 0$, then $x_T \to 0$ and $y_T \to \frac{h}{2}$ meaning $T \to E$. Also, if $t \to b$, then $x_T \to \frac{b+c}{2}$ and $y_T \to 0$ meaning $T \to F$. Thus, the locus of the point T is the EF line segment (without its extremities).

5. Comments

The GeoGebra file created above facilitates the discovery of the geometric shape that could be the locus we are looking for: by moving point M until it gets to point B we see that point T becomes point F. Likewise, if M becomes A, then T becomes E.

The property described in the above problem is independent of the given *ABC* triangle's shape. Convince yourself by moving any of the A, B, C points. Verify the calculations made for different cases by studying the algebra window: this way you can emphasize the connection between the geometric representations and the algebraic calculations.