

Correlating 3D Experiments with Dynamic Figures Analysis

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ABSTRACT. *Integrating geometry, algebra and calculus in computer-mediated activities by means of a free software such as GeoGebra is a challenge for all participants involved, and requires a reciprocal conceptual exploratory alternative for the teaching and learning of mathematics at pre-university level. This approach allows for the development of meta-mathematical competences in technical fields at introductory levels in sciences and enhances "projecting competences".*

1. A Dynamic Scenario of Mathematical Exploration

Synthesis: Integrating geometry, algebra and calculus in an effective solving problems environment represents a didactical and disciplinary challenge for teachers and students. It provides an experimental alternative for reciprocal exploration of concepts in learning and teaching mathematics. It harnesses the proto-mathematical power of spatial intuition into meta-mathematical means used in technology and applications.

This combination of perspectives allows the teacher to demonstrate, in front of students and together with them, strategies revealing the "behaviour" of figures. Both GeoGebra to overcome trial and error approach and Geometria, as a synthetic tool to define, solve and review those problems, are used.

Introduction - Our goal is the designing of classroom scenarios using open-source software, promoting collaborative effort in a wiki-like environment. GeoGebra allows an approach both experimental and conceptual, where the exposition of ideas, modelling and problem-solving is also the focus.

We believe that adequately integrating such approaches into projects can lead to the development of 'meta-mathematical' skills, which are particularly useful in technological instruction and technology applications.

The 3D aspect is handled, in a quasi-empirical manner, using the Geometria open-source software, which helps to manipulate solids by means of a tool set, with which one can measure, draw, cut with planar sections, combine and transform the objects, view and evaluate the results at various angles and in motion. This dual approach allows the user to conduct a heuristic search for solutions, by guided attempts and mutual validation of the models.

First, the problem is formulated in the Geometria environment to get a holistic three-dimensional perception of what the global goal is. Second, the analytical models built with GeoGebra are used to find a dynamic set of relationships, then the three-dimensional models, to which the results of the analysis are applied and readily assessed. As a next step of this presentation, we are going to show how the approach outlined above is applied to a concrete problem, from the analysis stage to the complete solution.

2. Case Study

Let us consider an example in which a sequence of queries is made, from descriptive to the explanatory, similar to the approach taken in didactic engineering¹. The case study will be conducted following the steps below:

- Start with a proposition....
 - Consider a solid in a simulated 3D environment and the relationship between perimeters, sections and volumes of its parts, from which data necessary to delimit the effective position can be derived.
 - Rediscover and analyze the relations by devising various representations using GeoGebra, by switching from one scenario to the other - allowing the two models to reciprocally corroborate and complement each other in order to align them into one consistent tool set.
 - Given a solid consisting of two triangular pyramids and a prism joined at their bases, with the prism in the middle, hereafter named Tal, the challenge is to cut it with a planar section of a given perimeter, within a certain tolerance percentage, so that the volume of the smaller cut-off is a given fraction ($1/7$, to start with) of the entire solid and, additionally, a relationship between relevant areas is required, which will be detailed at a later stage..
 - Split the problem into smaller tasks and establish the necessary framework for resolving it:
- After a preliminary analysis of the challenge, its meaning and possible approaches to addressing it, we proceed to establishing the conditions and requirements for extracting the necessary data. In the example above, we will first require the section plane to be parallel to the bases of the two pyramids and the cut-off volume to be $1/7$ of Tal's volume. That will help us to find a reference perimeter.
- Model the sub-problem above and solve it in Geometria environment.
- Analyze the relationship between the height of the cut-off, on the one hand, and the section perimeter, its area, as well as the cut-off volume, on the other hand. It is of utmost importance that the volume of the cut-off varies smoothly in function of the cut-off's height, due to the fact the section planes are required to be parallel to

1 Explanations on the “didactical engineering” theoretical framework and research methodology are summed up in the guides mentioned in the bibliography.

Tal's base. Depending on the level of the attending students, planes parallel to another fixed plane can be considered, which requires analyzing what changes in the relationship above will occur when the section plane passes through a vertex of the base.

- After establishing the perimeter of the section, we proceed to the analysis of other triangles of the same perimeter and determine which of them can actually be interpreted as sections of Tal.

- Analyze the emerging activity

- The first sub-problem deals with the relations between the height, the perimeter, the area of the section and the volume of the object.
- Those relations are studied from the geometric and the analytical points of view.

We use Geometria to test the first attempts and GeoGebra to develop a proper model to find out those relations. We consider the construction a “model” as it is much more useful to set them and to study them than a usual analytical figure. As the class is trying to solve the problem in the 3D environment, the students are surprised at the failure of their first spontaneous attempt – cutting the solid at $1/7$ of its height.

That is where GeoGebra kicks in. The students must develop a new interpretation of the problem and a more reflexive approach to figure out how to solve it. They are led to reformulate what they already know in formal terms about the relations between the height, the perimeter, the area of the section and the volume. They review their formal knowledge relevant to the problem at hand and become conscious of the actual relationship between height, on the one hand, and perimeter, area and volume, on the other hand, of which the first one alone is linear, while the other two are, respectively, quadratic and cubic. Using these findings, the students go back to the problem and try to find a new strategy which combines the practical approach and the formal recovered knowledge.

It is worth noting that the analytical phase is not a starting point, but the last resort, when the failure of the first simple attempt becomes obvious. The first attempt is not abandoned by students without reluctance, perhaps, precisely because of its simplicity. At times, they voice exclamations like, 'But, of course! You have to take the cubic root because...!', after they succeed to make a connection with the relevant part of their formal knowledge. Other exclamations betray the amazement of some students that a piece of formal knowledge could be useful in a practical context: 'Oh! Look, now I remember about the edge as the cubic root of the volume and all those formulas...'

This proves how difficult it is to make use of knowledge detached from practice and stored away in the stagnant repertoire of the scholarly erudition.

We can take this point even further, and note the superior agility of the students that have not only the skill set to do estimation and calculation but also the discipline of practice. Traditionally, intuition is held as something quite separate from formal knowledge and use of tools. What we observed, however, is that the teachers' approach to solving the problem is made smoother, more agile and faster by formal knowledge and use of technology that has been practiced so often that it has become part of her intuitive skill set –that is, the teacher is now able to “think with the software” in the way a student might draw with the pencil. In the end, both teacher and students approach the problem and rely on intuition, only the students' intuition can still be enriched with familiarity with the technology, experience with resolving problems and the discipline of practice. Perhaps, if

the students realize that their failures are partly a result of lack of practice, it can become a motivating drive for them to embark on the discipline of drill and practice, deeper study and sustained effort to improve in any field.

In addition, over time most teachers have been made to feel that they must choose between the didactic approaches of drill and practice and those based on problem-solving heuristic methods. This experience has shown that these two approaches are not only not diametrically opposed but should be used together to provide the best results. It is very difficult to be creative if you don't have the base and the practice in the basic skills. In this, mathematics students should take a page from athletes or artists, who improve talent and creativity by consistently practicing the basic skills that in turn support that space to get creative. In the classroom, we observe this in what we call the "creative paradox" –teachers get the best results by giving space to both "flights of fancy" and to disciplined practice.

Whatever the case, the mission of the teacher would be to revitalize the assimilation of that knowledge by applying it to the resolution of a problem, after helping the students assume the ownership of the said problem, hoping that the results will be transparent enough to successfully complete the validation process. Geometria, with its quasi-empirical environment and built-in teacher control, allows the students to readily connect the geometric models with the available algebraic tools, which provides additional validation.

The effort to make the students aware of the pitfalls of the *plain solution* (or *linear solution*, in this case) involves complex teaching tasks, with teacher's rewards coming only after they have been fully implemented.

We believe that the situation above calls for a teaching approach more nuanced than a simple explanation. The teacher's task is further complicated by the fact that the quadratic and cubic nature of dependence between the height and, respectively, the section area and the cutoff volume has been interiorized by her and appears to her the only thing that makes sense. Not so for her students, apparently. That is why it is important that the teacher listen to her students, understand the misconception at the origin of their initial mistake and walk them to the correct solution, through a series of steps that switch from Geometria to GeoGebra and back. After all, the necessary formal knowledge is already there, it is just that the students fail to *adequately* connect it with the practical problem at hand.

The emphasis above is to stress the fact that the students do have some experience of applying formal knowledge to practical situations, a straightforward and therefore *inadequate* one, in this case. This makes for a perfect opportunity to teach them the power of the nuanced approach. The attraction of versatile tools, such as GeoGebra and Geometria, is that it makes the teacher's task easier.

That brings us to the next sub-problem: finding a Tal section that has the same perimeter, a given area and, perhaps, provides for a greater cutoff volume.

The triangles of given perimeter form a dynamic set which is best analyzed with GeoGebra, with its rich dynamic modeling capabilities.

At this stage, Geometria is used as source of information, where the 3D scene is observed and the parameters measured, while GeoGebra serves as an analytical laboratory to combine and tweak the parameters, then evaluate the changes and find trends that might lead to a solution. The findings are then to be fed back into Geometria, at a later stage.

The GeoGebra model is built by the teacher, with students actively cooperating, aiming at the following tasks:

- Promote cross-questioning

- Define a solution methodology that can be reused in a different context
- Classify the problem by style and solution methodology.

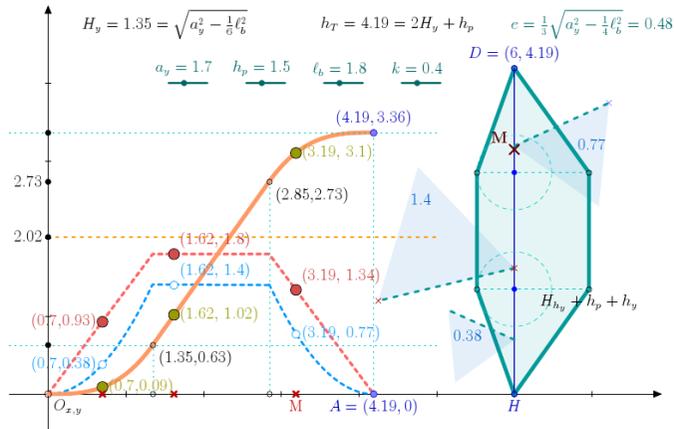


Figure 1: Modelling Tal 3D loci of perimeter, area and volume with GeoGebra

- Define a family of problems that can be solved using similar meta-mathematical procedures, based on the analysis figure, representative geometric constructions, the particular ways to explore the initial 3D scenario, the kind of constructions that are to be made in Geometria, and the kind of algebraic model created in GeoGebra.
- Create variations of the problem by changing the parameter values or the required relations. For example, require that the section area to be a k fraction of the one resulting from the first trial. Or the cutoff volume to be as large as possible.
- Explore different and even unexpected perspectives, which arise while building and exploring the simulation model.

In practical terms, the GeoGebra model allows to calculate the height of the cutoff as a function of the analysis figure. The value of the model transcends the immediate gain of solving the problem at hand. The model teaches how to represent a problem in an environment different from the one where it initially emerged.

The students learn how to extract the information that leads to the construction of a meaningful model and how to copy data between the two environments, by means of a proper interpretation of the parameters that are to be copied.

GeoGebra reinterprets the parameters harvested from Geometria's 3D scene in a way that allows us to assume that more than just a model is being created. It is, in fact, a simulation platform for solving the problem.

We are talking here, in Herbert Simon's terms, of simulation as a source of new knowledge. Some of this knowledge comes from the graphical representation of the relations between the measured parameters as a dynamic locus. Each point on the locus bears the value of a function. One explores that value seeking to minimize or maximize the function or bring the value close to the sought one..

- Analyze the emerging activity

It bears emphasizing that the teacher leads the development in front of the students and together with them.

She shares with them the rationales and techniques of constructing the model and defining the analysis figure.

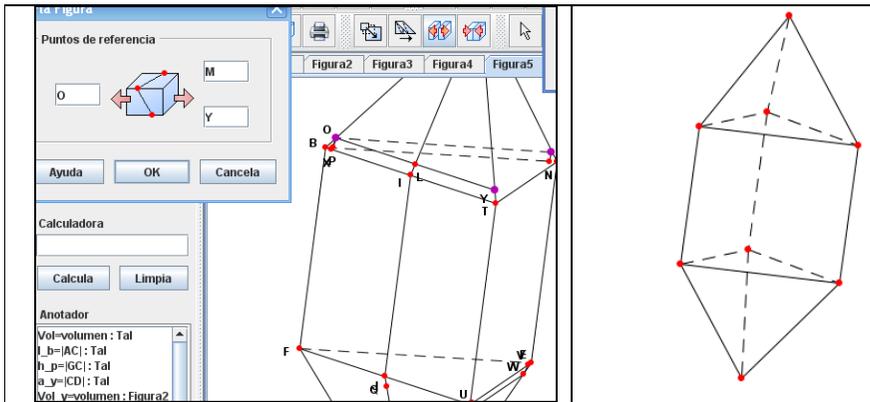


Figure 2: Tal in 3D Environment

Her meta-communicates not just an approach to solving a particular problem. Rather, she develops a larger context for solving such problems. She encourages the students' own interpretations, sets the rules for communicating them and arbiters between the ones that conflict with each other. She shows them how to make contrasting interpretations drive the solving process.

Simon [Simon1973] defines the above techniques as *projecting*.

In a broad sense of the word, projecting is distinguishing and emphasizing in a problem the underlying engineering perspective as opposed to the mathematical apparatus used for its modelling.

Accordingly, the teacher conducts the problem solving activity by raising issues that go beyond the mathematical means necessary to build the model.

Students learn not only how things are, but how they might be.

They learn how to articulate a solution design.

In what follows, we are going to describe the skills involved in formulating and solving the problem and outline the cooperative attitudes that come into play during the resolution of the problem and each sub-problem.

Development - Case study: posing a question in an unusual way

The challenge is formulated: *How could we find the triangles of a given perimeter and an area which is k times the maximum possible area, where $0 < k < 1$?*

Such an intentionally ambiguous assignment is at odds with the pedagogical contract (the tacit learning agreements)².

It is quite common that the teachers are trying to minimize the chances of their students' giving a wrong answer or making any mistake.

² See more about didactic contract (tacit academic learning agreement) in [Brousseau1988] and [Filloux1974].

They propose concepts, such as pedagogical contract, from a general perspective, i.e. not related to the specificity of knowledge that gives rise to interaction. They mention even ampler agreements, such as the one related to the concept of *consuetude* and, moreover, the concept of field configured by the *habitus* (see [Bourdieu1972]).

The rationale behind their choice, apparently, is that if mistakes and wrong answers are systematically voiced in the classroom, it will be more difficult to get the students *used* to correct answers, methodologies and procedures. As a result, the teacher favours an incremental approach, whereby the questions that minimize the risk of a wrong answer - the easier questions- come first. Once they are answered, slightly more difficult questions are asked, and so forth, toward deeper topics and more difficult questions.

The teacher's satisfaction is guaranteed as long as she asks a question only after making sure most of her students will be able to answer it.

This method also guarantees that no time will be wasted on explaining the questions per se. By the time any question is asked, the students will have already be prepared for it and readily produce the correct answer.

This conventional approach sacrifices the economy of knowledge in favour of a smooth classroom management, what could be, therefore, ultimately counterproductive. Having spent some time in this classroom environment where each question is easy to answer, indeed is almost rhetoric, the students come to build an expectation that the teacher will *never* ask a question that is too difficult, unclear or equivocal.

They will feel betrayed if that happens one day and they will feel hopeless and unprepared when the school years are over and the challenges presented by practical situations where their mathematical knowledge is required, are anything but clear and unequivocal.

The conventional approach rests on a general assumption that "every assignment should be perfectly clear, reachable and familiar and have one correct answer, with a subsequent explanation aiming to summarize the newly acquired concept without wasting much time".

This explains why the students collectively fail the challenge presented by the Tal problem.

They are acting in full conviction that the solution must be simple and within immediate reach, otherwise the teacher wouldn't have proposed the problem in the first place. In reality, it turns out that their 'solution' is incorrect. Luckily, the teacher does not have to communicate that to them, it is left to the Geometria's built-in answer evaluation capability.

After the first reaction of dismay and feeling betrayed by the teacher, the students may come to realize that they are facing a practical problem that needs to be solved. As any practical problem, it is self-contained. There is data that can be extracted using simulated tools. There are no prepared questions of increasing difficulty or a learned strategy that they are obliged to follow. Any strategy is good as long as it leads to a correct solution.

The assignment that forms the basis of this paper, on the contrary, creates a situation of uncertainty and leads to a series of exchanges, some of them irate. It opens the possibilities of negotiation³.

The students that normally tend to ignore the explanations now demand them. Their demands are not met with contempt, their voices are heard and their questions are welcomed⁴.

Thus, the dialog replaces the silent acceptance of the assignment, which is common in the conventional approach⁵.

3 [Godino2004] examines this issue conceptually in a series of successive papers.

4 A design of dialog-facilitating assignments are developed in *Fichas y Notas* [Brousseau1988].

5 [Young 1993] provides a critical description of the communication phenomena that occur in teaching situations associated with a distribution of roles between the actors, that is students and teachers, their voices and silences. [Chevallard 1997] analyzes its implicit attributes. Some of it can be found in *Fichas* mentioned below.

Attempting to Draw the Analysis Figure with no Numerical Data Available

The problem in question does not expose any data, at least not numerical data. The students have to *assume*, as opposed to *be given* data.

They must accept k as a variable parameter and they must assume the responsibility of finding the triangle of the greatest area and the conditions under which that would be achieved.

The negotiation is conducted using a crucial para-mathematical and/or meta-mathematical resource, whereby the dynamic figure of analysis becomes a medium in which the assignment can be performed and, likely, debated, collectively, under the teacher's supervision⁶.

The drawing is outlined, and then it evolves from a dynamic sketch into a model. (a model in the sense that it is representative and has the greatest degree of generality possible⁷.)

Dynamic Representation of Triangle Based on Inequations

The drawing is outlined using the software.

The drawing must be both specific and general enough to extend its scope⁸. First, together, the teacher and students draw a segment of a length equal to the perimeter of the first-trial triangle, which has been copied from the 3D solution. The segment is assigned a dynamically adjustable length.

The left end of the segment will coincide with vertex A of the triangle. The problem is reduced to locating the other two vertices of the triangle.

A first mathematical discussion, which was not apparent at the time the problem was formulated, ensues: *How do we draw in the absence of given values? What length do we use?*

This kind of a question, seldom posed by teachers, requires the students to distance themselves from an entrenched habit of drawing representative figures based on concrete dimensions.

This habit does owe to the quality of their means: sensory perceptions on the one hand and on the other hand, arithmetical information (not yet algebraic). In contrast, the question is posed in an algebraic way: it is an algebraic question. It requires modelling right from the beginning of drawing. Besides, it drives us to deal with conditions accomplished by a number of infinite triangles and by quite diverse, although not arbitrary, triangles.

Points B and C_E are usually placed arbitrarily and then we draw the two circles, centred at B and A with radii BC_E and $C_E P_{\text{erim}}$ respectively, and we find the third vertex – C – at their intersection. With all three vertices A, B and C set, we draw the triangle.

We vary the location of vertices B and point C_E and observe the changes of the triangle itself and the modifications in its position.

These changes include regular disappearances of the triangle.

6 [Legrand1993] discusses the conditions for a genuine classroom debate. This and other situations of classroom exchange are described in the recommended *Fichas de Clase* [Saidon2001].

7 The 'meta-mathematical' generally circulates in an implicit way. It can involve methods, structures, organization and principles.

8 See the official GeoGebra manual [Saidon2001-2009] for considerations on ways of representations.

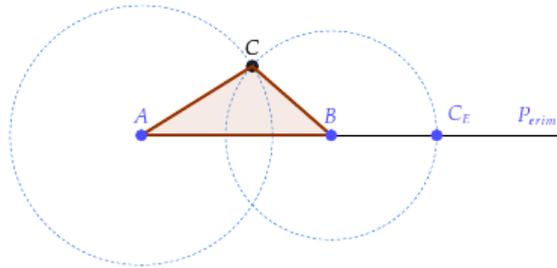


Figure 3: Sketching possible triangles of the same perimeter

The triangle vanishes when point B moves beyond the half-perimeter point – a virtual bound in segment AP_{erim} . The triangle vanishes also when point C_E is located so that BC_E or $P_{\text{erim}}C_E$ exceeds the semi-perimeter.

However bizarre it may seem to the students, the 'behaviour' of the triangle, including its disappearance, can actually be predicted based on their knowledge of elementary geometry. Thus, the behaviour of the analysis figure seems bizarre in spite of being perfectly deducible from basic mathematical facts.

However, students seem unable to apply in this context what they have been taught at a theoretical level, that is, the conditions of existence of a triangle. They seem unable to associate a theoretical object with this situation.

They are surprised at or passively neglect some bizarre episodes of the triangle behaviour, including its disappearance, as they seem unable to instrument the representative inequalities that form the conditions of existence of the triangle.

They are definitely unable to reformulate those inequalities in terms of semi-perimeter.⁹

Dynamic Trial

The students proceed to a first dynamic exploration of the drawing, with the explicit purpose of finding the triangle of the greatest area. Their endeavor meets a first obstacle in that the triangle occasionally vanishes.

This observation is often underestimated and even neglected by the students, even those of them who have solid mathematical training.

They neglect it, perhaps, because they are so concentrated on their task, which is to find the triangle with the greatest area. This lack of appropriate reaction on the part of the students should make the teacher think.

9 [Douady1986] analyzes the tool-object dialectics. The complementary object-tool dialectics, also observed in our case, is studied in [Piaget1989].

Haven't they noticed the disappearance and subsequent reappearance of the triangle? The teacher will seek to provoke the students' reaction by helping them with thoughtful interventions¹⁰.

A Surprising Practical Assessment of Perception Theories

We observed repeatedly that the disappearance of the triangle was being overlooked by the students.

Baffled as we are by this phenomenon, we record their reactions and persist in their repeating the exploration over and over until they become aware that something unusual is going on.

We noted that only when repeated exploration led to the phenomenon reoccurring on a frequent basis, did the students become aware of it. When that finally happens, the students accept the disappearance of the triangle as a new problem and the necessity of finding an explanation.

The students' failure to react deserves few more considerations. Their lack of reaction did not meet our expectations.

The didactic expectation was that the students would notice the disappearance of the triangle right away and attempt an explanation.

However, it turned out that the phenomenon may not be observable at once, although they would be able to integrate it later on. Perhaps, they accept the disappearance of the triangle when they can no longer *evade* it. Perhaps, the passive watching of the triangle disappearing is a pre-conjecture of sorts.

Indeed, while conjectures may appear in various situations, seldom does a conjecture arise on the basis of simple observation, even successive observations. Furthermore, not all elements of a dynamic construction have an easily observable behavior for a casual spectator. While the power of observation itself varies from individual to individual, what is noticed or what determines an observable phenomenon is linked to discernible intelligibility. The difference of perspectives could be attributed to the difference between casual observation and causal expectation.

Intentional observation implies more than "looking at" the results of a maneuver, calling for active "looking for" those results. This kind of inspection is the turning point between casual observation and the causal examination able to provoke conjecturing.

As obvious as the phenomenon appears to the teacher, it is unlikely that the students see it in the same way.

They fail to create spontaneous conjectures to explain what they are supposed to have seen as they are unable to note it: it is not an observable phenomenon for them as they are not intentionally watching.

The students are more likely to notice the effects when they are acting on a related purpose. In that case, they can not only observe a result, but even discern a pattern in the results. In other words, actions that are organized to achieve a certain objective are accompanied by a careful observation of *what* is happening and *when* it is happening.

The best results are achieved when the actions respond to adequately formulated challenges like 'What would you do to achieve this or that?'

10 The relation between the results of research in didactics and teacher's performance in class are vividly on display in such situations. See [Brousseau2002].

In our case, the students are concentrated on chasing the triangle with the greatest area. They are watching the variations of that area.

They are trying to see when the triangle has the greatest area and under what conditions. Later on, they might be trying to register the conditions which produce deviations from the maximum area.

That experience can be more revealing as it raises methodical inferences. A question arises, whether the way things are recorded is more or less related to the purpose –that is searching the greatest area triangle–.

The search criteria could be driving the actions of the students and vice versa, the results of their actions could be re-organizing the search criteria.

At the first stage, the actions aim at finding a triangle of the greatest area and the possible ways of achieving that by means of simple visual inspection of the triangle, whose variable shape is driven by sliding points B and C_E .

They may choose to simply compare each new shape with the previous one. The disappearance of the triangle may not be observable at this stage since it is not a phenomenon worthy of observation.

Geometry: From Visual Concepts toward Algebraic Models

The students fail to apply their theoretical knowledge of the inequalities between the sides of a triangle to explain the disappearance of the triangle.

When they finally recognize the disappearance of the triangle, they fail to accurately describe it. They fail to make a connection between the disappearance of the triangle and relative positions of points B and C_E .

The task of applying the algebraic in-equations to limiting the position of points B and C_E is anything but trivial.

The pedagogical contract has already been transgressed when the teacher posed a problem without numerical data to start the drawing with. Another transgression occurred because the problem contained open questions.

A third violation happened because themes and content became interrelated. Usually, these are considered separately, in a watertight structure of 'applications', where concepts, definitions, deductions and applications are organized as separate objects.

The problems tend to be organized as successive unrelated exercises that require answering a specific question each. In both the 3D and the dynamic environment of the analysis figure, the static figures approach gives way to a new approach based on an amalgam of topics that belong to different mathematical disciplines.

Crisis and Simulation

The critical situation depicted above is dealt using meta-cognitive competence, which plays a decisive role in the applications of mathematics.

A competence that is at the core of mathematical activity at a certain level: the ability to synthesize techniques, technologies and theories in order to form a *tool*.

A question arises, what do the results of our dynamic tests tell us about the relations? Relations we indeed could have anticipated as they obey the basic geometric facts.

The answer is not trivial and is related to results on simulation reported in [Simon1973].

This is a dynamic model of what algebraic manipulations usually provide, which is a set of relations used to analyze the given system by means of calculus.

The simulation performed on an appropriate model could be considered, from this point of view, as calculus in action.

The relations of the second and third order that are revealed during the process can establish a new starting point for further derivations of dynamic trials.

It not only makes possible simulations based on the geometric relations of the initial algebraic model, but also finding earlier unknown links from the designer's point of view. These analytical trials of third degree may be dynamically developed in order to:

- Consider appropriate loci drawn in GeoGebra's graphical view to graphically evaluate the recently found relations, aiming at a representation that facilitates, in terms of Duval, the conquest of the object.
- Find the functions (linear, quadratic, polynomial, exponential...) that fit best each of the drawn geometric loci
- Identify the functions to be interpreted at a later time using the full arsenal of calculus and GeoGebra.

A whole set of mathematical skills are being developed while the teacher and her students work toward the resolution of the problem whose model is being developed.

This situation facilitates a peculiar formative framework. Formative more than educational, because of its hands-on style involved in professional practices and institutions of professional performance.

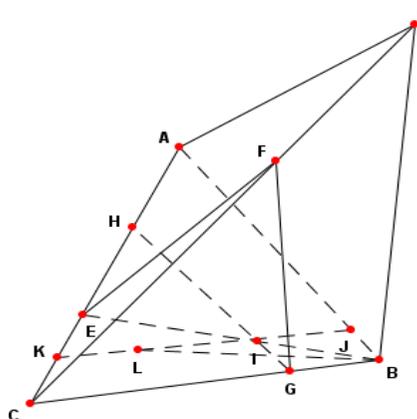


Figure 5: Pyramid Cut-Off

Design Problems

It is characteristic of various types of problems concerning systems of elements is that the relations and patterns of operation of the elements are known, the challenge is to predict how those systems are going to behave when the relations are changed dynamically.

Without going into great detail on possible derivations, let us study a problem as an example of a didactic situation and subject organization, of 'projectual' mathematics:

Problem Features: A Case of Projectual Mathematics

A software utility that enables the dynamic modeling of the problem as an open question, the representation and the analysis, opens several doors at once. There are tools instrumental during both the resolution of the problems and the design of the problems. This double instrumentation requires us to rethink the scope of the posed questions.

This mathematical and didactical reflection should take in account the increased range of what we are able to solve with such new tools.

We should reflect even with more emphasis in the wider scope and new quality of what we shall be able to question and cope with as we become more comfortable with these tools.

Quite often, the immediate application of new tools for teaching purposes is the dynamic explanation and illustration of a math subject, as the students are invited to explore the drawings and extract some data in the process.

Alternatively, the teachers choose an illustrative dynamic explanation of a subject, whereupon the students are asked to review the subject in that dynamic context.

Without questioning the value of such approaches, we are wondering if the technologies integrated in the professional, teaching and research practices, could aspire to be more than a privileged illustrative resource.

By advancing collaborative development, the model could be used as a dynamic test bench.

The problems could be outlined, be posed and solved, as shown in our case, to develop 'metamathematical' skills on the basis of curricular mathematics. Starting with:

1. Simple exploration. 'Move and see what happens.'

At this stage, understanding of what should be done and, simultaneously, non-understanding of the relations that would allow it can be observed.

2. Intermediate Exploration Level.

The goals to be achieved are clearer, but the means are still linked to the trials with partial successes or failures that are not always understood.

At this level, the students can answer questions such as 'How?.'

They are also able to answer and elaborate on questions like 'Let us change the points' locations and mix up everything on purpose to see if we could come back to this situation. Do you believe that is possible? Do you want to write it down before we proceed?'; 'Now that you have found a way out of this, do you think it is the only way?'; 'Could you teach your colleague how to do it without using the mouse directly?'

Questions like this could intersperse the teacher's interventions. Others like 'How could you prove that your goal has been achieved?' bring us to the next level.

3 Experimentation Level

- Instrumental, as anticipations of the course of action are created
- Modelizing, as control indicators are conceived and set.

'Projectual' features can be distinguished in this process, whereby the students provoke a trial, then analyze its results.

At first, the result is often erroneous.

Gradually, the students assume more responsibility for the outcome of each attempt as causal relationship between what is done and what is the result observed.

The 'projectual' activity integrates tasks and techniques that allow to define what methods would not achieve the purpose as opposed to those which would. This contrast allows making a clear distinction between the functional relations involved.

Methodologies

At each stage of the process, one can distinguish tasks that involve certain conjectures.

We may realize what conjectures the students make based on their actions. Ruling out one conjecture may facilitate the emergence of another conjecture, enriched by the constructive rescue of what failed to explain the result.

The teacher institutionalizes the monitoring and recording of what is, and what is not, related to the task at hand.

She shows the ways this methodology can be useful in other contexts¹¹. This kind of a positive assessment (not just 'learning by errors') of tasks, techniques and methodologies, to define the scope and rule out conjectures has little tradition in school despite its implicit recognition in academic, professional and research practice.

Stepping up the Tasks: Changing the Character of the Tasks

We have achieved substantial progress in the representation of the problem.

The students had to reformulate the conditions under which the triangle exists as function of perimeter, or rather half-perimeter.

Their experimentation involved the study of a mathematical subject and object such as Tal.

This dialectical counterposition characteristic of the tool-object¹² duality will reoccur during the problem resolution.

From Models to Simulations

Diagnostic indicators are needed to find out how the construction *operates*.

Reliable indicators are required to guide the trials during the entire process, to consider the dimensions of the triangle and the volume evaluated in complementary representations.

At a certain stage of the resolution, the purpose of the exploration is find trends in variation of the triangle's area, such as increases, decreases, cancellations, maximum values, etc. This is task of a different order.

Evaluating trends in a measured parameter variation is a calculus task par excellence. Moreover, inasmuch as the model's ability to evaluate the variation trends is concerned, we are performing an 'intra-mathematical' simulation.

The Loci Exploration Experience

GeoGebra can plot the locus of vertex C of the triangle of a constant base and a variable

11 'Institutionalization' in terms we use it here, is defined in [Brousseau1994].

12 [Douady1986] deliberates on this tool-object dialectic relationship in addition to showing the potential of the inter-play between various mathematical disciplines.

ration between the other two sides.

The teacher must show the (easy) technique and explain what a geometric locus is.

The resulting curve looks familiar. It looks like an ellipse and is likely to be one. Besides, the values of the height and the area vary correspondingly.

GeoGebra is useful in establishing if the curve is indeed an ellipse.

The teacher explains how to compare the curve with the conic passing through five of its points.

It turns out that the resulting conic and the locus overlap in a way that the students are not able to distinguish one from the other. Since the two objects are of different order, the Boolean comparison is improper and useless.

The two objects must be studied separately. To study the geometric locus, on the one hand, and the equation and representation of the ellipse, on the other hand, the students learn the proper techniques and their theoretical background.

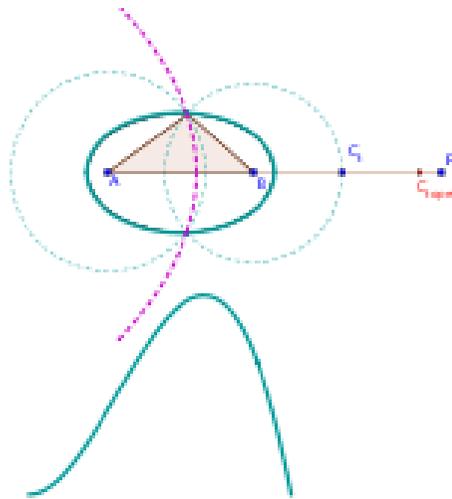


Figure 6: Model Sketch

Experimenting with the Formulation

To solve the problem, it is necessary to relate the ratio between the triangle's sides and its maximum area.

We need to find for which member of the ellipse set the greatest area of the triangle is achieved. GeoGebra enables this correlation, which is shown in Figure 6, by linking the drawing with the recorded data set and the appropriate formulas, such as the Heron formula. By combining various representations, we come to a result which can be solved entirely with calculus tools.

To solve the problem, it is possible to explore and observe the triangle set dynamically, and then select the position that best suits the task. A precise formulation is not indispensable.

Such an open answer poses deeper mathematical questions. We leave those questions to our audience, as we believe that the best results can be achieved in a collaborative exchange, hopefully sparked by the presentation.

Conclusions

Below is a summary of the remarkable points in our experience

- Clarifying a problem, however simple it appears to be, requires cooperation
- Interpreting the problem requires an opinion exchange and a teacher-guided debate
- Drawing a representative sketch helps to reveal relations
- By exploring the behavior of the construction, an exploration of causal relationships is started
 - By linking the construction's dynamic behavior with algebraic formulas and graphic representation, calculus was invoked
 - After examining the sketch as a foundation for inferences and trials, it became a model in terms of dynamic simulation
 - To study the model, approaches from various mathematical disciplines were intertwined.
 - To revise the generality of the model, its limits and scope had to be validated
 - To gain deeper comprehension, successive conjectures have to be evaluated
 - To find a correct solution, the arbitrary "trial and error" method has to be overcome and a methodology of systematic essays and trials recollection must be taught and employed, Diverse mathematical competences were invoked during the resolution of this problem. It required an amalgam of know-how, formal knowledge and practical skills.

The last stage of the problem can be regarded as a case of control leading to the search of the locus that verifies the conditions.

From the design perspective, we consider the building of the test bench central to the entire task. The test bench provides means for conceptual and experimental mathematical study in a 'projectual' style.

We call it projectual because its relations and conditions correlate our intentions with what is being accomplished.

The ensuing activity gives birth to a model of practices assumed by the teacher in front of her students and the 'projectual' quality of the solution. The model developed with the software allows us to juxtapose what has been projected with the results obtained. The solving methodology implies a reflection on what is being built – in the interaction between the subject and the object – and, simultaneously, controlled.

We perform tasks that require conceptual and instrumental skills. We deal with a model and the simulation of the model behavior, while developing high level mathematical competences, which can be useful in applications.

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