

# GeoGebra Tool for Building Conceptual Understanding in Mathematics

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**ABSTRACT.** *At the beginning of the 3<sup>rd</sup> millennium we are facing dramatic changes in the basic nature of teaching and learning strategies caused by massive use of new ICT. We can benefit from this development in general, and in mathematics especially, as currently available dynamic and visual learning environments as software GeoGebra could affect our perspective in terms of the content and comprehension of mathematics education. Few ideas are presented on how GeoGebra can be used as tool for creating cognitive connections between different representations of mathematical concepts, which form the necessary background for better conceptual understanding, steady knowledge and mathematical literacy.*

## 1. Introduction

Research in cognitive psychology indicates that our brains store knowledge using both words and images. Instruction that targets and engages both of these systems of representation has been shown to significantly increase students' comprehension and retention. Explicitly engaging students in the creation and usage of non-linguistic representations has even been shown to stimulate and increase activity in the brain. Mathematics is cognitive process-thinking that requires the dual coding of imagery and language. Imagery is fundamental to the process of thinking with numbers and conceptualizing their functions and their logic. The Greek philosopher Plato said, "And do you not know also that although they [mathematicians] make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble, they are really seeking to behold the things themselves, which can be seen only with the eye of the mind?"

Mathematics is the essence of cognition. It is thinking (dual coding) with numbers, imagery and language; reading/spelling is thinking with letters, imagery and language. Both processes, often mirror images of each other, require the integration of language and imagery to understand the fundamentals and then apply them. Dual coding in math, just as in reading, requires two aspects of imagery: symbol/numeral imagery (parts/details) and

concept imagery (whole/gestalt). Mathematical skill requires the ability to get the gestalt, see the big picture, in order to understand the process underlying mathematical logic. The ability to create mental representations for mathematical concepts is directly related to success in mathematical reasoning and computation, as stated in [BT03].

Manipulatives are concrete or symbolic artefacts that students interact with while learning new topics. They are powerful instructional aids because they enable active, hands-on exploration of abstract concepts. Research has shown that computer-based manipulatives are even more effective than ones involving physical objects, in part because they can dynamically link multiple representations together.

At the beginning of the 3rd millennium we are facing a dramatic change in the basic nature of teaching and learning strategies caused by the massive use of new technology. We can benefit from this development in general, and in mathematics especially, as currently available dynamic and visual learning environments could affect our perspective in terms of the content and comprehension of mathematics education as well. There is an increasing importance of dynamic linking of multiple representations in facilitating students' visualization because students can explore, solve, and communicate mathematical concepts in various ways, such as using dynamic multiple representations and mathematical modelling. Simply showing pictures or figures is not sufficient to encourage students to visualize or use various representations. Extensive usage of new communication technologies has enlarged the pool of cognitive tools and possibilities of their application in teaching and learning processes with visible consequences and feasible improvements.

## 2. Cognitive tools

Cognitive tools have their irreplaceable position in the didactics and educational theory and there exist a lot of ideas on how we can understand their role in education. Cognitive tools allow users to explore mathematical concepts dynamically and have been increasingly discussed in the recent years, as can be seen in many papers appearing in journals on mathematical education. Authors Luis Moreno-Armella, Stephen J. Hegedus, and James J. Kaput provide an historical overview of symbolization, which can be considered as a kind of the evolution of cognitive tools. They identify five stages of the evolution of symbolization "from static, inert inscriptions to dynamic objects or diagrams that are constructible, manipulable and interactive", in [MHK08], p. 103.

The "static inert stage" of symbolization in mathematics consists of a wide spectrum of textbooks and handouts prepared by teachers for ages and still widely used. The second stage denoted as "static kinaesthetic/aesthetic stage", dwells in the use of chalk and marker pens on black-green-white boards and slides with overhead projectors and allows users to erase and change the inscription. Besides the flexibility to make changes, these tools also provide opportunities to use several colours to emphasize particular notations. In the third "static computational stage" users get a static presentation of the input they provide to some certain devices such as calculators and graphing tools. Gabriel Salomon [Sal93] favours this type of interaction with cognitive tools because he believes that users can have more opportunities to deal with high order thinking tasks if they leave low level cognitive activities to the tools. The last two stages of this evolution framework focus on the dynamic perspectives of symbolization. In the fourth, "discrete dynamic stage" a discrete co-action between user and environment is described. The users have more control over the process and the output obtained from the media, because they have the ability to change and

manipulate the input dynamically through the first interactive tools that have recently appear in some maths software solutions. In the fifth stage called “continuous dynamic stage” users can be interacting with the real objects through dynamic interactive devices and therefore they can get instantaneous and continuous feedback.

The dynamic worksheets prepared in GeoGebra software enable users to create dynamic mathematical objects such as graphs and functions and to interact with these objects. Assuming that these mathematical objects are the real objects of this platform, although the platform itself is virtual in nature, then the continuous interaction between these objects and users could be considered as a continuous dynamic interaction. Similarly, GeoGebra can be considered as a haptic device detecting the movements of the slider and adjustable objects described in this platform, as mentioned in [KG09].

### 3. Some features of GeoGebra as manipulative tools

GeoGebra is an intuitively controllable application suitable for all users without any extra mastership in computer literacy. It can serve for development of instructional materials in mathematics in many different forms, types and styles, and for all levels of mathematical education. Downloadable free from its website, while nothing but a Java 6 platform is necessary for its full operation, GeoGebra seems to be particularly easy and intuitive to obtain, use and learn. GeoGebra user interface offers a rich graphics menu for drawing various objects, while the complete construction protocol is saved and it appears in any chosen language from the 45 available, therefore no translation is necessary and free sharing of developed instructional materials is genuinely supported all over the world. Users are encouraged to visit GeoGebra webpage and GeoGebra user’s forum GeoGebraWiki, a free pool of teaching materials for this dynamic mathematics software where everyone can contribute and upload materials.

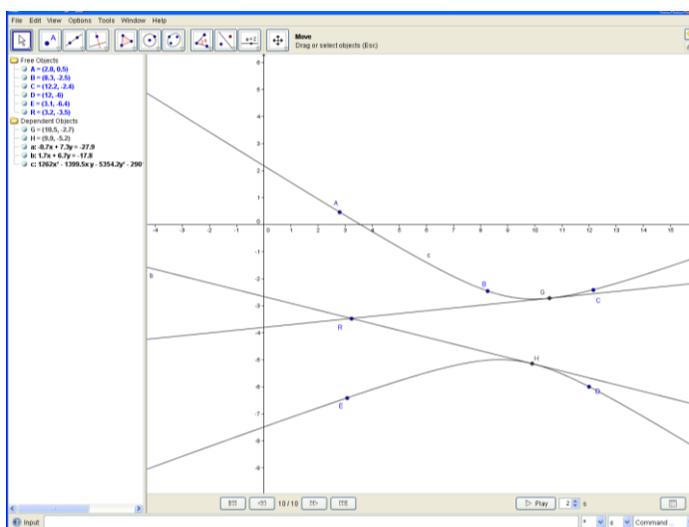


Figure 1. GeoGebra user interface

GeoGebra has a built-in Cartesian coordinate system, and accepts both geometric commands (drawing a line through two given points, a conic section determined in different ways, in Figure 1. by 5 points) and algebraic ones (drawing a curve with a given equation). Among its more interesting features is the ability to draw tangent lines to algebraic and even transcendental curves at given points, while equation of this tangent line is available immediately too. This double representation: the geometric – synthetic one and the algebraic-analytic one is one of the greatest advantages of GeoGebra software that mostly suits to didactic aims of full comprehension of basic mathematical concepts. One can enter the objects either as geometric objects (via drop down menus) or as algebraic objects – pairs of coordinates, functions – via the entry line. Moving the objects in the Geometry window (see Figure 1.) changes the expressions in the Algebra window accordingly (on the left in Figure 1.), and vice-versa, editing the expressions in the Algebra Window results in the respective change in the Geometry Window. This is the main feature of GeoGebra meeting the demands of many didactics and educators to provide for the students as many representation forms as possible.

Teachers can guide students to understand how these two representations of mathematical objects are connected to each other. Changes in the superposition of the conic section determining points are reflected in the form of its equation. Following these changes in the algebraic window students can realise the role of the coefficients of the bi-quadratic form. Existence of the tangent line to the conic section passing through the chosen point can be investigated by changing its position, while equations of the tangent lines appear immediately in the updated form.

In this way, students are engaged to study relations of mathematical objects in more complex form, in both ways, geometric - synthetic and algebraic - analytic representations receiving thus comprehensive knowledge. After having developed an understanding of the relation between the particular type of determined conic section and its equation, and tangent line to a conic section and its equation, students have the opportunities to create cognitive connections between their previous knowledge and the outcomes of the current exploration. They can create various cognitive connections, and by further explorations they can be engaged in asking what-if questions, as: What if would two of the conic section determining points coincide? What do I have to change in the 5 determining point's superposition in order to receive ellipse, parabola, or hyperbola? What if I move point R to be the point on the conic section or to be its interior point?

Many various questions that can be created by teachers and students are leading to heuristic learning approaches, hands-on exploration of abstract concepts and cognitive connections. GeoGebra represents in this sense the example of a cognitive tool of the fifth, continuous dynamic stage, as users are really interacting with the real objects through dynamic interactive devices and they receive instantaneous and continuous feedback in two simultaneous forms.

Teachers can prepare suitable applications for demonstration of particular mathematical concepts, and students can work directly with these supported materials. But they are also encouraged to create their own applications, developing thus deeper insight and comprehensive cognitive connections, and cultivating their creativity and imagination. They can also contribute this way to the development of new variegated instructional materials and worksheets for the overall usage and share in the classroom. Consequences of this approach are extremely important and significant, as the usage of dynamic cognitive

tools influences the former passive role of students in the process of education. Now they become actively involved in the process, acting not only as self-learners, but also in the role of demonstrators for other students in the classroom.

GeoGebra separates mathematical objects into free objects and dependent objects. Where the dependent objects are defined by an explicit construction (algebraic or geometric) the construction steps can be encapsulated into a tool. Once the tool has been defined a new button appears on the tool bar and a corresponding function name is available to the user. Such tools are essentially functions, and may operate with geometric objects such as circles, lines and points. The tools function within GeoGebra is interesting because it allows geometrical functions to be defined, which illustrate a key mathematical process: encapsulation or compression. Using these tools it is possible to extend the software in natural ways, just as mathematical domains are extended during normal teaching. The idea of a tool as a function appears in [San08]. It allows the user to behave mathematically in two important ways: compression and extension. Compression or encapsulation is the process by which a mathematician takes a complex procedure or construction and represents it by a single step. Simple example of compression is arithmetic multiplication: repeated addition is compressed to a single step. Geometrical example of compression is a theorem of Euclidean Geometry that given three points which are not-collinear a unique circle can be determined through them. This multi-step construction, which requires drawing perpendicular bisectors to line segments, can be encapsulated into a single operation. GeoGebra of course provides a button to do so. For mathematicians it is interesting to notice that this compression construction process results in a well defined geometrical function: take three points and return a circle (or a straight line).

Users are allowed to create their own tools as macro-constructions, while buttons for calling these geometric functions are immediately available in the basic GeoGebra menu. This is one of the features that are extremely convenient in the view of customers, who are free in customisation of the software interface menus according to their needs.

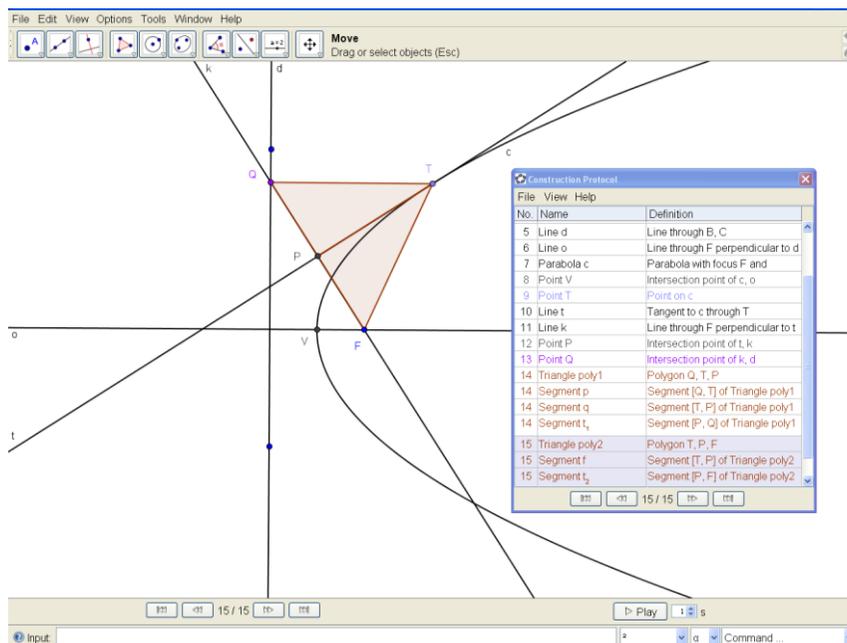
#### **4. Dynamic constructions and applications for on-line usage**

Dynamic constructions are one of the most powerful GeoGebra features. Users are able to create dynamic worksheets demonstrating certain properties and features generally applicable to specific basic geometric objects and dynamically change the form of these objects. Properties of tangent lines to parabola and position of points symmetric to the parabola focus with respect to all parabola tangent lines are illustrated in Figure 2. Construction is stored in a construction protocol, available on request from the view menu, and can be followed step by step in there. Construction protocol appears in the chosen language of the GeoGebra interface that leads to no need of its translation to other languages. In addition, the whole construction can also be re-played via the navigation bar for construction steps in step-by-step mode, while the choice of automatic play is available with the offer of customisable timing set up.

Other example from mathematics can be the explanation of the concept of a function derivative and its geometric meaning. This basic concept from the mathematical analysis is very often underestimated as rather easy to understand and visualise. Anyhow, dynamic illustration of the changing slope of the tangent line to the graph of a function of one real variable according to the value of the function first derivative in the respective point is rather crucial for the overall understanding of the function behaviour. It is easy to

illustrate intervals on which function is defined and to estimate intervals, where function is increasing (positive slope) or decreasing (negative slope), and to find points, in which the first derivative equals to zero, therefore function is reaching the extremes in there, its maximum or minimum value. Consequently, tangent line in the point of extreme appears as parallel to the coordinate axis  $x$ . Also points, in which the first derivative of the function is not defined, can be illustrated also, as the slope of tangent line to the function graph in such point goes to infinity and tangent line in this point to the function graph is perpendicular to coordinate axis  $x$ . The concept of the global extremes and maximal and minimal value of a function of one variable on the given interval can be investigated clearly and dynamically as well.

Apart from the standalone application, GeoGebra also allows the creation of interactive web pages with embedded java applets. These targeted learning and demonstration environments are freely shared by mathematics educators on collaborative online platforms like the GeoGebraWiki, part of the International GeoGebra institute that has been established to coordinating the work of thousands of volunteers all over the world in the structure of accredited national GeoGebra institutes in different countries.

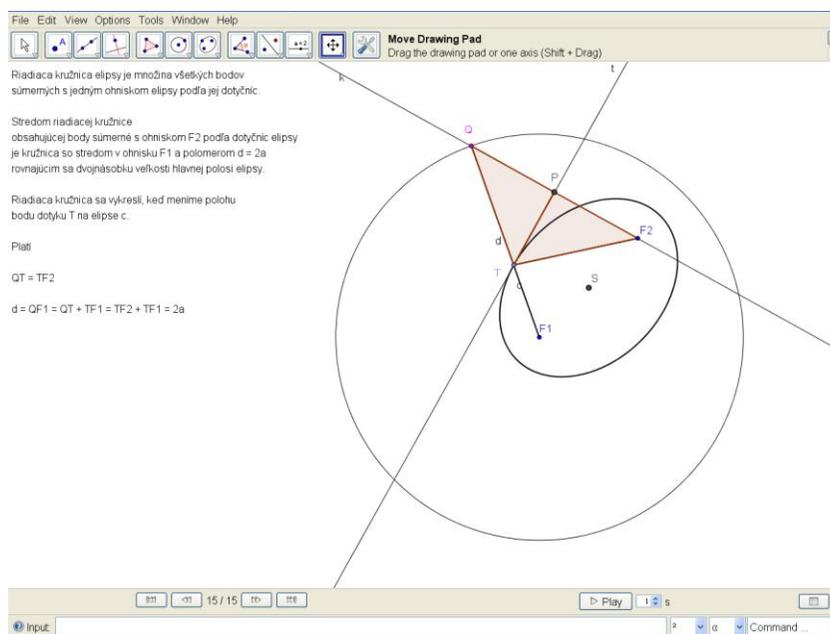


**Figure 2.** GeoGebra dynamic construction with Construction protocol

GeoGebra dynamic spreadsheets enable to produce interactive presentations directly on the web serving as electronic instructional sources for e-learning solutions in the form of dynamic cognitive tools. These html pages can be used directly from Internet and are presentable in all common web-browsers, while there is no need to install GeoGebra software in the user's computer. The only necessary prerequisite in addition to the web-browser is the installed Java support that is essential. Illustration in Figure 3. is from the Slovak database of e-learning materials at the GeoGebra Institute of Slovakia, at the

webpage <http://geogebra.ssgg.sk/>. Properties of the ellipse control circle are presented here, while the geometric function locus is used here to demonstrate the form of the set of all points that are symmetric to one focus of an ellipse with respect to its tangent lines. Moving the tangent point  $T$  on the ellipse, symmetric points  $Q$  appear and draw the control circle. According to the specific setting chosen for the export of the source GeoGebra file ggb into the on-line presentable html file, button for construction protocol, navigation bar for construction steps and play button with timing set-up window can be visible in the dynamic spreadsheet application. The entire construction can be then replayed and followed step-by-step. Interactive approach on the dynamic spreadsheet can then be used for animation of the tangent point movement along the ellipse, while determined control circle is generated as locus of symmetric points  $Q$ .

A collection of dynamic instructional materials is available also in the European Virtual Laboratory of Mathematics, in English from the EVLM Central Portal, in the Central EVLM database at the address [www.evml.stuba.sk/databasemenu/menu\\_files/frame.htm](http://www.evml.stuba.sk/databasemenu/menu_files/frame.htm), and in Slovak language from the Slovak EVLM portal, in the Slovak EVLM database at the address <http://slovak.evml.stuba.sk/elearning/>. Many of applets presented in these databases enable users to act interactively and to directly manipulate received results using sliders.



**Figure 3.** Dynamic GeoGebra web application, construction of ellipse control circle

Scope and position of the sliders are determined by applet authors but might be kept free; therefore user can move them freely in the drawing worksheet. These enable smooth change of shaping parameters and can be set to automatic change, creating thus a real animation available on web. Manipulated by users themselves, these sliders serve for exploring relations between different parameters of any object and finding their role in the

determination of the shape of its representation and visualisation as geometric figure, analytic expression, equation, mathematical formula, or any logic relation and statistical table, in [Ve09].

Collection of dynamic applets created in GeoGebra as module “Cycloidal curves” is extensively utilized by students at the Faculty of Mechanical Engineering, Slovak University of Technology in Bratislava, in both regular and distance learning classes. Module consists of theoretical part introducing different kind of trochoidal curves, their synthetic determination as trajectories of a point moving in two simultaneous revolutions about given centres. This movement can be simply interpreted as a movement of a circle rolling without slipping around a fixed circle. Curves are visualized in GeoGebra applets, where students can be freely exploring influence of separate parameters to the final form of the dynamically manipulated trochoidal curves, and to see their geometric properties and equations. Three available sliders enable users to change parameters for the number of epitrochoidal curve knots and cusps in the case of epicycloidal curve, distance of the moving point from one centre of revolution (centre of the moving circle) and distance of the two centres of revolutions (centres of moving and fixed circles) in Figure 4. Applet may contain few explanatory words that introduce the illustrated object, while user is free to manipulate sliders and investigate their role in determining the modelled object shape. Hypocycloidal curves might be studied in applet illustrated in Figure 5., where a Steiner hypocycloid is created as trajectory of a point attached to the circle rolling inside a fixed circle.

Students from distance learning classes particularly appreciate this interactive approach and availability of all materials free on-line, where they are accessible directly without any special requirement to the user computer.

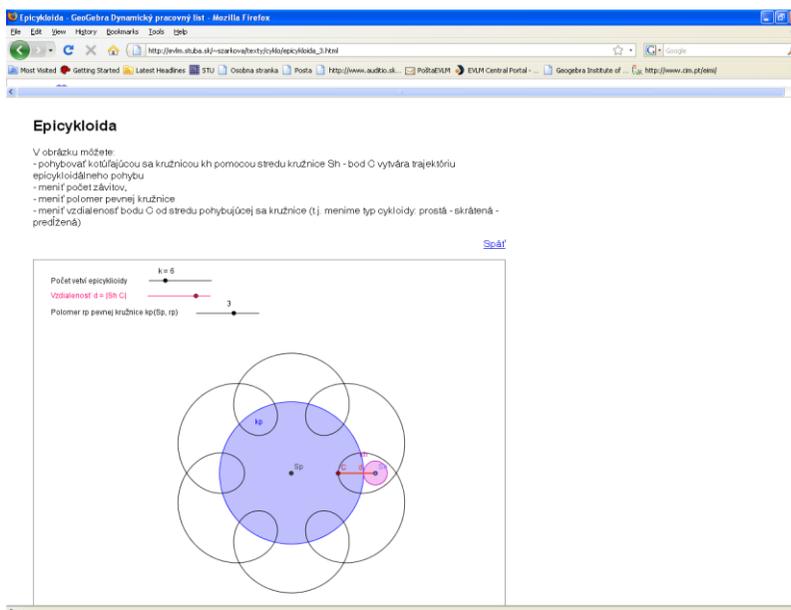


Figure 4. Dynamic epitrochoid

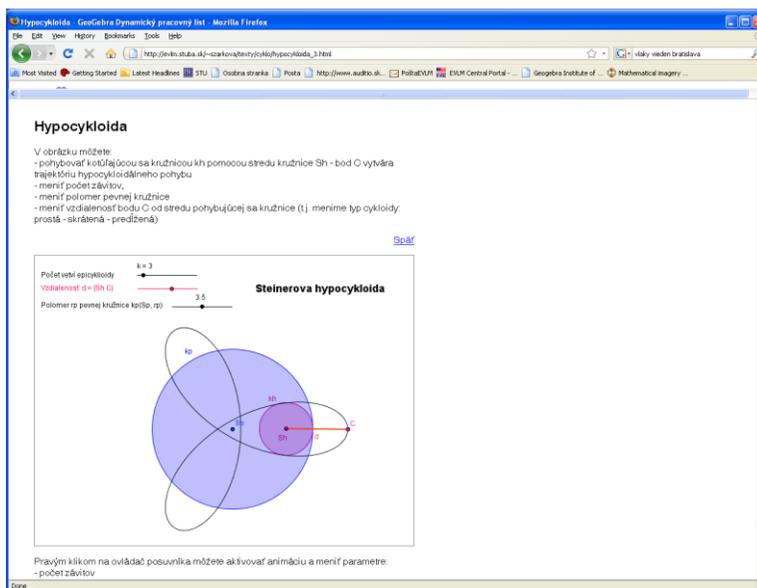


Figure 5. Dynamic Steiner hypocycloid

The available possibility in GeoGebra to export files as applets in the form of html files is an extremely important feature and a strong advantage. This feature puts GeoGebra to a much favourite position among other software solutions, where user must have more deep knowledge of the software to receive the same effect. Some exceptions might be considered with respect to webMathematica java applets – jsp files and Mapplets developed in Maple. Anyhow, these applications do not provide so many manipulative features as GeoGebra solution, as for instance development of a dynamic drawing workplace, which is uniquely possible in GeoGebra only, without any problems. Prospective authors using other software solutions for development of dynamic materials must in addition by the respective commercial software products that are relatively quite expensive. Certain technical problems and compatibilities still remain unsolved.

## 5. Conclusions

Concerning the software development of GeoGebra, authors are always looking for good ideas for new features and extensions. With the recent addition of statistics charts, commands, and new tools, while authors are also working on introducing 3D geometry in the similar dynamic mode as it currently works for the case of 2D, it will be crucial to focus on keeping the software's user interface simple and easy. Thus, in the future, we can expect to have a universal tool for math education combining Dynamic Geometry, Computer Algebra, semantics of mathematical formulas, dynamic 2D drawing worksheet and possibly 3D Dynamic Geometry that will hopefully keep its simplicity and user friendliness.

Finally, it should be mentioned that simple drawing of mathematical objects and figures is not enough for the building of a comprehensive understanding of basic mathematical concepts. On the other hand, creative dynamic activities are essential also to

the development of one's technological content knowledge. This seems to be consistent with the general notion of mathematical understanding as one's growing competence to navigate through various representations of mathematics concepts illustrated dynamically. GeoGebra seems to be a didactic tool supporting these efforts to a high level, and in an easy, natural and user friendly way. These characteristics do predetermine this free-ware application to be used in teaching, learning and exploring mathematics into such a large extent as possible. Not surprisingly to the benefits and satisfaction of all groups of its users, teachers and students including.

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