A Comparative Study of the Effects of Using Dynamic Mathematics Software on Students' Understandings about Tangent Concept

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Abstract

The concept of tangent is an important concept in understanding many mathematics and science topics. Earlier studies that focused on students' understandings of the concept of tangent reported that students have various misunderstandings and experience difficulties in transferring their knowledge about tangent line from Euclidean geometry to calculus. This study aimed to examine the effects of using Dynamic Mathematics Software in the teaching of tangent concept on students' understandings and their generalization types. The study concluded that utilizing Dynamic Geometry Software in the teaching of tangent concept help students overcome difficulties and reconstruct their existing knowledge regarding tangent concept. In the light of the results some suggestions are made for improving the teaching of tangent concept.

Keywords: *Tangent concept-Dynamic mathematics software-Generalization in Mathematics-Prospective mathematics teachers*

1 Introduction

High school and university mathematics curricula include some common concepts that are dealt more comprehensively and in detail at the upper level. In these cases the concept in question either becomes applicable to a wider class of mathematical entities or is re-defined in a different and more formal way. The concept of function is an example for the first situation. First, the concept of function is introduced to students as an operation that maps numbers to numbers. Later on, the inputs and outputs of a function can be different kinds of elements, as in determinant function. As to the second situation, the definition of tangent to a curve in calculus differs from that of in Euclidean geometry. In such cases students are expected to generalize their knowledge in line with the formal theory. However, the results of many studies in advanced mathematics literature show that this is not a straightforward process, and that students experience difficulties in reconstructing their knowledge of a particular concept (Tall, 1988; Biza, 2007; Vinner, 1982).

Students encounter with the tangent concept in three different contexts during their mathematics education. Students' first encounter with the concept of tangent takes place in Euclidean geometry then they revisit it in the context of analytic geometry and finally they define it in a different and more formal manner in calculus. Tall (1988) argues that in such cases students' mental imagery of a concept may grow in a way that is not coherent and consistent, and previous experiences may colour the concept when it is met in new contexts. Research (Vinner, 1982; Tall, 1987) showed that for some students, early experiences of the tangent concept in Euclidean geometry result in a belief that the tangent is a line that touches the graph at one point and does not cross it. This, in turn, causes students to form a concept image that brings about cognitive conflict when drawing tangent to different types of curves, such as drawing tangent to a curve at an inflection point.

1.1 Students' Understandings of Tangent Concept

The concept of tangent is an important concept in understanding many mathematics and science topics such as geometric meaning of derivative, curve sketching, interpreting problems involving rate of change, understanding direction fields and their role in the solution of differential equations, applications in other sciences, etc. Earlier studies that focused on the concept of tangent reported that students have various misunderstandings and experience difficulties in transferring their knowledge about tangent line from Euclidean geometry to calculus. Tall (1987) investigated the effect of using interactive computer software on students' understanding of the tangent concept. Tall defines a term named "generic concept" as one abstracted as being common to whole class of previous experiences. Following this, he describes the notion "generic tangent" in which the tangent line touches the graph at exactly one point. Tall concludes that the experiences of the experimental group helped students develop a more coherent concept image with an enhanced ability to transfer their knowledge to a new context. Although the experimental groups in this study outperformed the control groups, Tall reported that a significant number of students retained the notion of "generic tangent". A study of Biza (2007), which involved 182 first year mathematics undergraduate students, focused on differences among students' conceptions about tangent line. This study attempted to model students' conceptions about the tangent line in terms of five characteristics regarding the relationship between the tangent line and the curve. The study revealed eight models of students' conceptions about tangent line but did not identify a hierarchy among the models. Each model possesses different mathematical insufficiencies such as rejecting a line as tangent when it coincides with the curve or splits the curve into different semi-planes. Biza, Christou, and Zachariades (2008) aimed to model high-school students' understandings about tangent line. Through the analysis of students' responses to the questionnaire used in the study, they divided students into three distinct groups each represent a basic perspective on tangents, and identified a hierarchy in students' abilities to complete the questionnaire tasks among the groups. In addition the study reported that some students experience difficulties in constructing and identifying tangent lines especially at inflection and edge points and that some students' decision in identifying a line as tangent depends on the way the curve is represented (symbolic or graphical). Another study that was conducted on in-service teachers by Potari *et al.* (2006) showed that not only students but also in-service teachers experience difficulties in understanding tangent concept. This study aimed to explore teachers' mathematical and pedagogical awareness in higher secondary education, specifically calculus teaching and the concept of derivative. The results of this study showed that the mathematical knowledge of the teachers, who participated in the study, about relationship between the tangent to a circle and to a curve is fragmented. The teachers were reported to think that the concept of tangent in the circle case is not the same as in a curve. Moreover the study reported that teachers had difficulty in seeing the tangent to a circle as a special case of the tangent to a curve. Taking into account these studies, it can be concluded that traditional approach to teaching the tangent concept does not suffice to provide students an in-depth understanding of the tangent concept. Therefore, there is a need to design learning environments which help students overcome the difficulties that they experience in generalizing their knowledge about tangent concept.

1.2 The Concept of Tangent in Turkish Educational Setting

In Turkey, students' first encounter with the concept of tangent takes place during the 7th grade of elementary education (age 13). At this time the concept of tangent introduced to students under the topic of "Positions of line and circle in the plane". The only think that students learn about tangent is that it has only one common point with the circle.

Students' next encounter with the tangent concept takes place in three different contexts during their secondary school education (Ages 15-17). First, students study the concept of tangent in the context of Euclidean geometry at the 10th grade. In this context students study the theorems for angles formed by the intersections of tangents and secants on and outside a circle. Besides students study the formulae that relates the side lengths of the segments that are formed by the intersection of a tangent and a secant of a circle. At the last year of high school students study the concept of tangent in both analytic geometry and calculus contexts (grade 11). In the analytic geometry context students learn how to find the equations of tangent lines that passes through a point that lies either on and outside of a circle or on an ellipse or on a hyperbola. In the calculus context students study the concept of tangent under the topic of derivative. Students learn how to find the equation of a tangent to a curve by differentiating the algebraic expression of the curve. Neither elementary nor high school mathematics curricula demands teachers to show examples of tangents that may contradict with the notion of *"generic tangent".* At the mathematics teacher education program students study the concept of tangent in the course named "Analysis I". In the book which is most widely used for this course around the country, the notion of tangent to a curve on a point is first demonstrated as the convergence of secant lines and then defined formally by means of derivative. The book gives an example in which a tangent is drawn to a curve at the inflection point. Besides the book treats vertical tangents (tangent at a cusp) as a special case and gives the following curve as an example (Fig. 1).

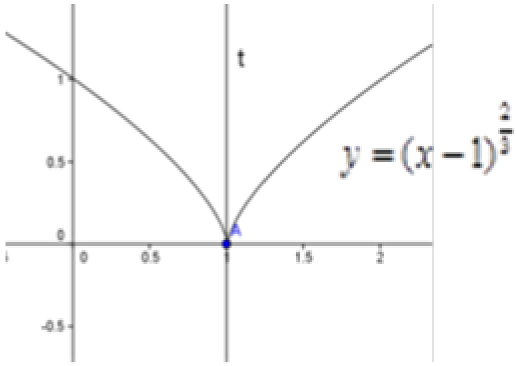


Fig. 1 Tangent at a cusp

2. Theoretical Background

In this study the term *concept image* is used to refer to a student's total knowledge regarding tangent concept. Tall and Vinner (1981) define the term *concept image* as the total cognitive structure that is associated with a particular concept, which includes all the mental pictures and associated properties and processes. A student's definition for a specific concept in his/her concept image may differ from the formal definition of the concept. Tall and Vinner (1981) consider this situation as a serious type of potential conflict factor and assert that it may impede the learning of formal theory.

In the expansive generalization a student expands the applicability range of his/her existing schema of a particular concept without reconstructing it (Harel & Tall, 1989). Moving from the definition, a student achieves expansive generalization about tangent concept when he/she adds new types of curves that do not contradict with the properties that he/she attributed to tangent in Euclidean geometry context, such as tangent line must have only one common point with the curve (Fig. 2). In the reconstructive generalization a student reconstructs his/her existing schema in order to widen its applicability range (Harel & Tall, 1989). In the case of tangent this type of generalization is expected to occur at the time when the concept of tangent is defined by means of derivative in calculus context. The difference between reconstructive and expansive generalization is that a reconstruction is essential for the former type while there is no reconstruction in the later. Therefore if a student achieves a reconstructive generalization process about tangent concept, his/her definition for the tangent is radically changed. Moreover students who achieved reconstructive generalization see the tangent to a circle, an ellipse, or a parabola is the special cases of the formal definition. As seen in Fig. 2 a tangent which intersects with the curve more than one point or drawn through a stationary point or coincides with the curve is acceptable which would be questionable in the context of Euclidean geometry. Moreover students who achieve reconstructive generalization are aware that there is no tangent to a curve at a cusp because they realize that the right and left derivatives are not equal at that point. On the contrary, those who fail to reconstruct their schema would accept a tangent at a cusp because that it touches the curve at only one point, that is to say the situation agrees with the *generic tangent*. Similar to reconstructive generalization, disjunctive generalization occurs when a student meet the same concept in a different context. But at this time student constructs a new, disjoint, schema to deal with the problems that he/she confronts in the new context (Harel & Tall, 1989). In the case of tangent, if a student fails to achieve a reconstructive generalization after the instruction in calculus context, then the student forms a disjoint schema about tangent concept to deal with the problems in this new context (Fig. 2). Therefore when a student with disjunctive generalization confronts with a tangent problem, the student first decides which context does the problem belong to and then he/she attempts to solve the problem by evoking the related part of his/her concept image. To make this clear, suppose that a student with disjunctive generalization was asked if there is a tangent to the curve of the function $f(x) = |x|$ through, its cusp, the point (0, 0). If the student was provided with only the curve of the function without its algebraic expression, then the student would evoke his/her Euclidean context dominant-schema and would draw a tangent through the cusp. On the contrary if the student was asked the same question but this time only provided with its algebraic equation, then the student would evoke his/her calculus context dominant-schema and would not draw any tangent by asserting that the left and right derivatives at that point are not equal.

3. The Purpose and Research Questions

Undoubtedly, it is teachers who lead students through the process in which students generalize their knowledge of tangent concept from Euclidean geometry to analysis. As mentioned, some studies in the literature reported that prospective teachers possess misconceptions about tangent concept. It is obvious that the mathematical experiences of prospective mathematics teachers during their university years are the final chance for them to develop a sufficient understanding about tangent concept before their professional career. The overall aim of the presented study is to examine the effect of the learning environment supported by dynamics mathematics software on students' understandings of the tangent concept. Specifically, this study seeks answers to the following questions;

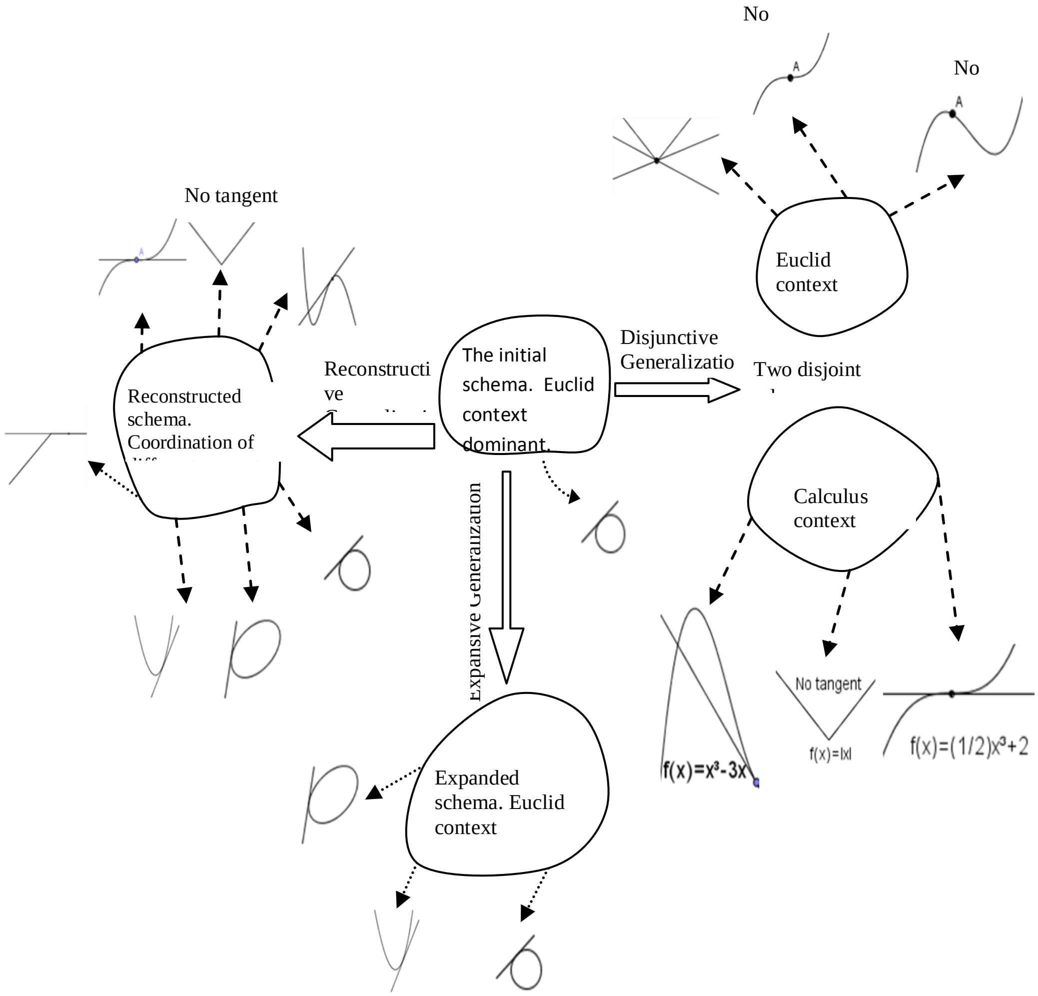


Fig. 2 The depiction of generalization processes with regard to the tangent concept

i) Does the use of the software have an effect on students' concept images with regard to dominant context?

ii) Does the use of the software have an effect on students' generalization processes?

iii) Is there a relationship between the type of generalization process and the dominant context?

4. Methodology

4.1 The Instructional Setting

The setting for the study was a course named Calculus-I offered to four classes of elementary mathematics education program at Fatih faculty of education in Karadeniz Technical University during the fall semester of 2010. At the beginning of the study two classes were randomly selected as control group. The control group consists of 67 students and was taught using a traditional approach without using computer software. In the control group students were taught the definition of the tangent to a curve through a point that lies on it by means of derivative. During the teaching session some questions were asked and their solutions were demonstrated in the class. These examples include tangents that pass through cusp, stationary, and inflection points. Moreover in the control group some curves to which a tangent cannot be drawn from the given point, for example $f(x) = |x|$ through the point (0, 0), were discussed.

The other two classes in experimental group (n=69) studied the tangent concept in an environment supported with the dynamic mathematics software *GeoGebra*. The teaching of the tangent concept for the experimental group took place in a computer laboratory where the students worked in pairs on computer. The intervention took place in two sessions each lasted approximately two hours. During the first session a worksheet was assigned to each pair of students in which the concept of tangent to a circle was considered. The content of the worksheet consists of an investigation phase of a slightly modified version of the Euclid's 16^{th} proposition included in the third book of Elements and questions about the activity. The students completed the tasks, included in the worksheet, in the computer environment. At the end of the first session a class discussion around the last question of the worksheet which asked students to define the tangent concept took place. The aim of this discussion was to convey to the students the notion that the tangent line is the limiting position of secant lines that pass through the tangent point and an arbitrary point on the circle as the arbitrary point approaches to the tangent point (Fig. 3).

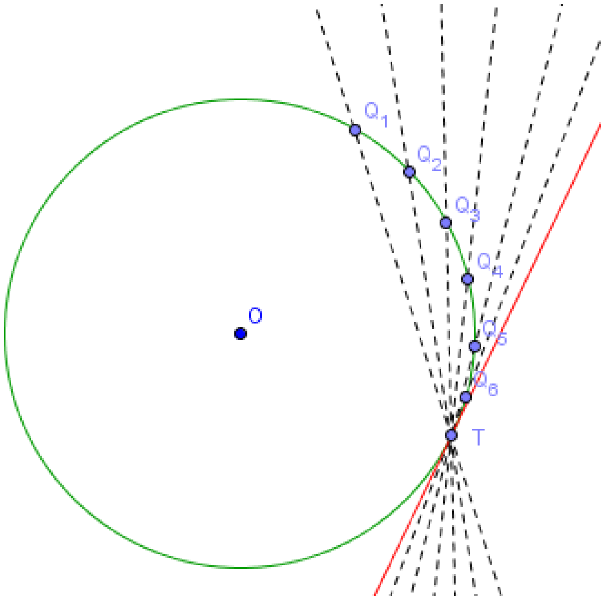


Fig. 3 Tangent as the limit position of secant lines

Briefly, with this activity we intended to enrich student's concept images of tangent concept with regard to its definition. After a week from the first session the second session took place, again in the computer laboratory. The aim of the second activity was to make students grasp the relation between the derivative value of a function at a point and the slope of the tangent line at that point. A worksheet which included the graph of sinus function and a fixed $A(x_0, f(x_0))$, and an arbitrary, subjected to an independent variable, $C(x_0+h, f(x_0+h))$ point was assigned to each pair of students (Fig. 4). Similar to the first activity, the students were asked to find an expression for the slope of the tangent as they make the arbitrary point closer to the tangent point. At the end of the session a class discussion during which the students presented their expressions was held. The session finished with the conclusion that the expression was "$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$", which is the derivative value of the function at $A(x_0, f(x_0))$.

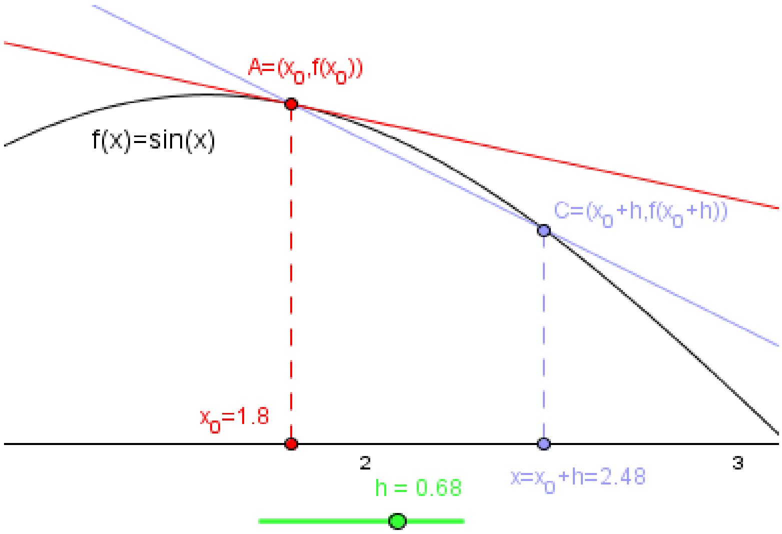


Fig. 4 a snapshot of the second activity

4.2 Data Tool

In this study, an open-ended test consisting of eleven questions is used as the data collection tool. We took into consideration the earlier studies in the literature (Tall D., 1987; Biza, Christou, & Zachariades, 2008), and the suggestions of the experts in the field in selecting the questions included in the questionnaire. At the beginning of the study the test was administered to students as a pre-test (Appendix 1). After the instruction the students took a slightly modified version of the same test as a post-test. The test consists of three sections: defining tangent concept, drawing tangent to curves, and finding the equation of tangent from algebraic data. In the first section we asked students to define the tangent concept while not restricting them to give a formal mathematical definition. In other words students were free to give a non-mathematical expression if they wish. In the second section of the test students were asked to draw tangents, if exists, to given curves through given points and also to provide justifications for their answers if they conclude there is no tangent through the specified point for a curve. In the selection of the curves, we took into account students' misconceptions reported in the previous studies. Hence the examples in this section asked students to draw tangents through a cusp, an inflection, a point where the tangent coincides with the curve, a point that causes the tangent intersect with the curve more than one point, and a point which lies on a parabola. In total there are five different curves with different aforementioned characteristics. In the last section the students were asked to find the equations of tangent line to the given functions through the given point. The functions in this section are presented only with their algebraic expression. Each of the curves in the second section is the corresponding geometric representation of one of the functions given in the last section, so there are five algebraically represented functions in the last section. In the post-test version of the test first section was same but the curves in the second section and their corresponding algebraic equations in the last section were changed while maintaining the characteristics of the curves.

4.3 Data Analysis

The students in both groups divided into two groups with regard to the context (Analysis, Euclidean) that dominates their concept images based on their responses to the first question. If a student used the derivative concept in his/her definition for the tangent concept then we arrived at the decision that the context which dominates the student's concept image was analysis. For example, the answer of a student who was classified in the analysis group was "*The tangent line to a curve through a point on the curve is a line whose slope is equal to the derivative value of the function at the tangency point*". On the contrary if a student did not use the concept of derivative in his/her definition, instead, tried to define tangent concept only by providing an example represented geometrically then we determined the context that dominates the student's concept image as Euclidean.

The questions in the other two sections were analysed together. These questions served to determine the type of generalization that each student achieved regarding tangent concept. A student's answers to a particular curve in the second section and its corresponding function in the last section were analysed together and coded as an ordered pair. The first element of the ordered pair indicates whether a student correctly identified the existence of tangent to a particular curve and, if exits correctly drew the tangent. Similarly the second element of the ordered pair indicates whether the student correctly identified the existence of the tangent to the corresponding function and, if exits correctly found the equation of the tangent. If a student's answer to a question was true then a "+" sign, on the contrary, a minus "-" sign was set for the corresponding element of the ordered pair. For example if a student drew an incorrect tangent or concluded that there was no tangent to one of the curves, whereas to which a tangent could be drawn but found the correct equation of the tangent from the corresponding algebraic expression of the curve in the last section then the student's answers to this pair of questions coded as (-,+). If a student did not provide an answer to a question then the corresponding element of the ordered pair left blank. So a student's answers to each pair of questions can result in nine different ways. If a student did not provide answers to two or more questions either in the second or the last section then the student was excluded from the analysis. Consequently a student's answers to the questions in the last two sections form a set of five ordered pairs. The number of students compatible for the analysis was 55 for the control and 60 for the experimental group in the pre-test while the numbers were, respectively, 54 and 57 in the post-test. If a student's answer set includes four or five consistent ordered pairs ((+, +)) then the student's generalization type was determined as reconstructive. On the contrary if a student's answer set includes four or five inconsistent ordered pairs ((+, -) or (-, +)) then the student's generalization type was determined as disjunctive. Moreover students' justifications to their answers in the second section revealed their misconceptions. The numbers of students who possess each misconception were calculated for each group. To answer the research questions chi-square tests were conducted on student's pre- and post-test results.

5. Findings

5.1 The effect of the use of software on students' concept images with regard to dominant context

The aim of the first question used in the open-ended test was to determine the context that dominates a student's concept image of tangent concept. The students were divided into two categories with respect to the context that dominates their concept images both prior to and after the study. Table 1 presents the frequencies of students belonging in each category based on their responses to the pre-test.

Table 1 Cross tabulation of context and group on pre-test

		Euclid geometry	Calculus	Total
Group	Control	51	16	67
	Experimental	56	13	69
Total		107	29	136

To determine whether there was a significant difference in the proportions of students between the experimental and control groups prior to the study a chi-square test of independence was conducted on students' pre-test results. The chi-square test result shows no significant difference between these group proportions, $X^2(1, N=136) = .515$, $p>.05$. Therefore the groups were concluded to be homogenous with respect to the dominant context prior to the study. After the instruction took place students were again divided into the categories based on their responses to the first question of the post-test. Table 2 presents the frequencies of students belonging in each category.

Table 2 Cross tabulation of context and group on post-test

		Euclid geometry	Calculus	Total
Group	Control	46	21	67
	Experimental	14	55	69
	Total	60	76	136

To determine whether there was a significant difference in the proportions of students between the experimental and control groups after the intervention a chi-square test of independence was conducted on students' post-test results. The chi-square test result shows a significant difference between these group proportions, $X^2(1, N=136) = 32.255$, $p<.001$. As seen in Table 2 the proportion of the number of students belonging to Euclid geometry to the number of students belonging to calculus in the control group is approximately equal to six times the proportion of the number of students belonging to Euclid geometry to the number of students belonging to calculus in the experimental group.

5.2 The effect of the use of software on students' generalization processes

The aim of the questions included in the second and last section of the test was to determine the type of (reconstructive, disjunctive) generalization processes of the students' regarding tangent concept. Moreover the justifications of students for the questions included in the second section of the test revealed their misconceptions about tangent concept. Table 3 presents the frequencies of students belonging in each category based on their responses on the pre-test.

Table 3. Cross tabulation of group and generalization type on pre-test

		Reconstructive	Disjunctive	Total
Group	Control	6	49	55
	Experimental	6	54	60
Total		12	103	115

To determine whether there was a significant difference in the proportions of students between the experimental and control groups prior to the study a chi-square test of independence was conducted on students' pre-test results. The chi-square test result shows no significant difference between these group proportions, $X^2(1, N=115) = .025, p>.05$. Therefore the groups were concluded to be homogenous with respect to the type of generalization process prior to the study. After the intervention students were again divided with respect to the type of generalization process. Table 4 presents the frequencies of students belonging in each category.

Table 4. Cross tabulation of group and generalization type on post-test

		Reconstructive	Disjunctive	Total
Group	Control	9	45	54
	Experimental	52	5	57
Total		61	50	111

To determine whether there was a significant difference in the proportions of students between the experimental and control groups after the intervention a chi-square test of independence was conducted on students' post-test results. The chi-square test result shows a significant difference between these group proportions, $X^2(1, N=111) = 59{,}300, p<.001$. As seen in Table 4 when compared to control group much more students in experimental group achieved reconstructive generalization regarding tangent concept.

In the second section of the test the students were asked to draw tangents, if exits, to the curves through the marked points and to provide justifications for their answers if they conclude that there is no tangent for a given curve through the marked point. These justifications revealed students' misconceptions regarding tangent concept. These misunderstandings and the frequencies of students that possess each of them are given in Table 5.

As seen in Table 5 for every misunderstanding, the drop in the number of students who possess a particular misunderstanding in the experimental group is bigger than that of in the control group. This finding suggests that the use of the software help students overcome their misunderstandings regarding tangent concept.

Table 5 Misunderstandings and the frequencies of the students

Misunderstandings	Pre-test Groups		Post-test Groups	
	Control	Experimental	Control	Experimental
The tangent line must have only one common point with the curve	35	30	28	9
The tangent line must not split the curve	17	8	14	4
The tangent line must not coincide with the curve	36	32	29	8
There is no tangent at an inflection point	25	22	19	7
Having only one common point with the curve is a sufficient condition for a line to be regarded as tangent	18	20	15	5

5.3 The relationship between the type of generalization process and the dominant context

The final aim of the present study is to determine if there is a relation between the type of a student's generalization process and the context that dominates the student's concept image of tangent concept. To answer the final question the students in both groups combined together and a chi-square test was performed on students' post-test results. Table 6 shows the frequencies of students belonging in each category. The chi-square test showed a significant relation between the type of generalization and the context, $X^2(1, N=111) = 28.56, p<.001$.

Table 6: Cross tabulation of dominant context and generalization type

		Generalization type		
		Reconstructive	Disjunctive	Total
Dominant context	Euclid geometry	12	36	48
	Calculus	49	14	63
Total		61	50	111

6. Conclusion and Discussion

The main question which we adressed in the present study was 'What is the effect of the learning environment supported by dynamics mathematics software on students' understandings of the tangent concept.' In the study we considered the understanding of tangent concept according to two dimensions: (a) the context that dominates a student's concept image of tangent; (b) the type of a student's generalization regarding tangent concept. As to the first dimension, the findings show that the use of the software in the teaching of tangent concept is more effective in releasing students' concept image from the constraints of Euclidean geometry than the traditional approach. The majority of the students in the experimental group became able to relate the derivative concept to the definition of tangent while the majority of the students in the control group could not establish this relation. As a result of the intervention the majority of the students in the experimental group generalized their knowledge of tangent in line with the formal theory and became able to determine the existence of tangent to a curve when the curve is represented either algebraically or geometrically. On the other hand the majority of the students in the control group could not achieve reconstructive generalization and as a result their decision for the existence of tangent for a specific situation was dependent on the representation of the curve. This finding emerged in our study is consistent with the study of Biza et al (2008). Although the learning environment supported with dynamic mathematics software helped the majority of the students gain a deeper understanding about tangent concept, there were still students in the experimental group retaining the notion of *generic tangent*. This situation also emerged in the study of Tall (1987). Though Tall (1987) did not mention any reason for this situation, in this study we think that this might be due to the short time during which students worked with computers.

6.1 Educational Implications

The students' justifications for their answers in the second section revealed five different misunderstandings regarding tangent concept. However if looked closely it can be concluded that they are same in nature. All these misunderstandings are the result of overemphasizing the notion of *generic tangent* in the teaching of tangent concept in Euclidean geometry courses. These misunderstandings also addressed in other studies (Biza, 2007; Tall, 1987; Biza, Christou, Zachariades, 2008,). The findings show that the learning environment supported with dynamic mathematics software is more effective in remedying these misunderstandings than the traditional approach. The final aim of the present study was to determine if there is a relationship between the context that dominates a student's concept image and generalization type. The findings show that there is relation between these two variables. Students whose concept image is dominated by calculus context were more likely to achieve reconstructive generalization.

References

Biza, I. (2007). Is the tangent line tangible? Students' intuitive ideas about tangent lines. In D. Kuchemann (Ed.), *Proceedings of the Conference of British Society for Research into the Learning of Mathematics*, *27(1)*, pp. 6-11. London.

Biza, I., Christou, C., & Zachariades, T. (2008). Student perspective on the relationship between a curve and its tangent in the transition from euclidean geometry to analysis. *Research in Mathematics Education,*doi: 10.1080/14794800801916457

Harel, G., & Tall, D. (1989). The General, the Abstract, and the Genetic in Advanced Mathematics. *For The Learning of Mathematics, 11*(1), 38-42.

Potari, D., Zachariades, T., Christou, C., Kyriazis, G., & Pitta-Pantazi D. (2006). Teachers' mathematical and pedagogical awareness in calculus teaching. In S. Alatorre, J. Cortina, M. Saiz, & A. Mendez, (Eds.) *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 2, pp. 846-848. Mexico*

Tall, D. (1987). Constructing the concept image of a tangent. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th International Conference of International Group for the Psychology of Mathematics Education*, *3*, pp. 69-75. Montreal.

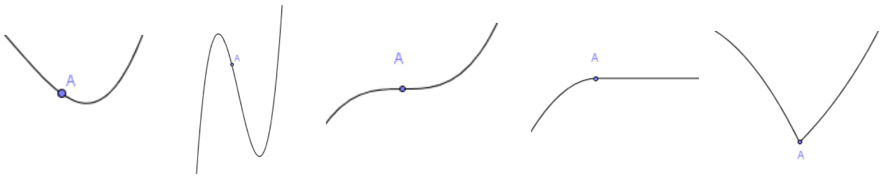
Tall, D. (1988). Concept image and concept definition. In J. d. Lange, & Michiel Doorman (Eds.), *Senior Secondary Mathematics Education* (pp. 37-41). OW&OC Utrecht.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Vinner, S. (1982). Conflicts between definitions and intuitions: The case of the tangent. In E. Vermandel (Ed.), *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education* (pp. 24-28). Antwerp, Belgium: Universitaire Instelling Antwerpen.

Appendix 1: Tangent Test

1. What is the definition of the tangent to a curve ? if you can not provide a mathematical definition try to define by your own words.
2. Draw tangent lines to the given curves, if exits, through the marked points. If you conclude there is no tangent to a curve please justify your answer.



3. Find the equations of tangent lines , if exits, through the specified points to the curves whose algebraic equations are given.

a) $f(x) = x^3 + 3$, A(0,3)

Equation: ....

b) $f(x) = \begin{cases} -x^2, x < 0 \\ 0, x \geq 0 \end{cases}$, A(0,0)

Equation: ....

c) $f(x) = \begin{cases} -x^2 + 3, x < 1 \\ e^{x-1} + 1, x \geq 1 \end{cases}$, A(1,2)

Equation: ....

d) $f(x)=x^2 +2x -4$, A(-2,-4)

Equation: ....

e) $f(x) = x^3-9x^2 +23x-15$, A(2,3)

Equation: ....